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GENERALIZED KÄHLER MANIFOLDS WITH SPLIT TANGENT BUNDLE

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Abstract. We study generalized Kähler manifolds for which the corresponding complex structures commute and classify completely the compact generalized Kähler four-manifolds for which the induced complex structures yield opposite orientations.

1. Introduction

The notion of a *generalized Kähler structure* was introduced and studied by the second author in [26], in the context of the theory of generalized geometric structures initiated by Hitchin in [28]. Recall that a generalized Kähler structure is a pair of commuting complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ on the vector bundle $TM \oplus T^*M$ over the smooth manifold M^{2m}, which are:

- integrable with respect to the (twisted) Courant bracket on $TM \oplus T^*M$,
- compatible with the natural inner-product $\langle \cdot, \cdot \rangle$ of signature $(2m, 2m)$ on $TM \oplus T^*M$,
- and such that the quadratic form $\langle \mathcal{J}_1 \cdot, \mathcal{J}_2 \cdot \rangle$ is definite on $TM \oplus T^*M$.

It turns out [26] that such a structure on $TM \oplus T^*M$ is equivalent to a triple (g, J_+, J_-) consisting of a Riemannian metric g and two integrable almost complex structures $J_\pm$ compatible with g, satisfying the integrability relations

$$d^c_+ F_+ + d^c_- F_- = 0, \; dd^c_\pm F_\pm = 0,$$

where $F_\pm = gJ_\pm$ are the fundamental 2-forms of the Hermitian structures $(g, J_\pm)$, and $d^c_\pm$ are the $i(\overline{\partial} - \partial)$ operators associated to the complex structures $J_\pm$. The closed 3-form $H = d^c_+ F_+ = -d^c_- F_-$ is called the *torsion* of the generalized Kähler structure.

These conditions on a pair of Hermitian structures were first described by Gates, Hull and Roček [21] as the general target space geometry for a $(2, 2)$ supersymmetric sigma model.

As a trivial example we can take a Kähler structure (g, J) on M and put $J_+ = J$, $J_- = \pm J$ to obtain a solution of the above equations. One can ask, more generally, the following

Question 1. When does a compact complex manifold (M, J) admit a generalized Kähler structure (g, J_+, J_-) with $J = J_+$?

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The case of interest is when $J_+ \neq \pm J_-$, i.e. when the generalized Kähler structure does not come from a genuine Kähler structure on (M, J). In this paper, we refer to such generalized Kähler structures as *non-trivial.*

Despite a growing number of explicit constructions [3, 12, 29, 33, 39], the general existence problem for non-trivial generalized Kähler structures remains open. On the other hand, there are a number of known obstructions, or conditions that the existence of a generalized Kähler structure imposes on the underlying complex manifold, which we now describe.

Firstly, it follows from the definition that for a complex manifold (M, J) to admit a compatible generalized Kähler structure it must also admit a Hermitian metric whose fundamental 2-form is $\partial\bar{\partial}$-closed. This condition on (M, J) is familiar in Hermitian geometry. It is trivially satisfied if (M, J) is of Kähler type (i.e. (M, J) admits a Kähler metric). When M is compact and four-dimensional ($m = 2$), a result of Gauduchon [22] affirms that any Hermitian conformal class contains a metric with $\partial\bar{\partial}$-closed fundamental form. Hermitian metrics with $\partial\bar{\partial}$-closed fundamental form naturally appear in the study of local index theory [10], on the moduli space of stable vector bundles [40], and have been much discussed in the physics literature where they are referred to as 'strong Kähler with torsion' structures. Complex manifolds admitting such Hermitian metrics are the subject of a number of other interesting results [18, 20, 25, 32, 47]. Examples from [18], together with the results of [19] and [20], show that there are compact complex manifolds of any dimension $2m > 4$ which do not admit any Hermitian metric with $\partial\bar{\partial}$-closed fundamental form.

Secondly, Hitchin [29] showed that if (M, J) carries a generalized Kähler structure (g, J_+, J_-, H) such that $J = J_+$ and J_+, J_- do not commute, then the commutator defines a *holomorphic Poisson structure* $\pi = [J_+, J_-]g^{-1}$ on (M, J). In the case when $H^0(M, \wedge^2 TM) = 0$, for instance, this result implies that for any compatible generalized Kähler structure on (M, J), the complex structures J_+ and J_- must commute, i.e. $J_+J_- = J_-J_+$.

Thus motivated, we study in this paper non-trivial generalized Kähler structures (g, J_+, J_-) for which J_+ and J_- commute. In this case $Q = J_+J_-$ is an involution of the tangent bundle TM, and thus gives rise to a splitting $TM = T_-M \oplus T_+M$ as a direct sum of the (± 1)-eigenspaces of Q. Our first result, Theorem 1, proves an assertion first made in [21], which can be stated as follows: *the sub-bundles $T_\pm M$ are tangent to the leaves of two transversal holomorphic foliations $\mathcal{F}_\pm$ on (M, J_+) and g restricts to each leaf to define a Kähler metric.*

The fact that $T_\pm M$ are both *holomorphic* and *integrable* sub-bundles of TM directly relates our existence problem to a conjecture by Beauville [8], which states that the holomorphic tangent bundle TM of a compact complex manifold (M, J) of Kähler type splits as the direct sum of two holomorphic integrable sub-bundles if and only if M is covered by the product of two complex manifolds $M_1 \times M_2$ on which the fundamental group of M acts *diagonally.* This conjecture has been confirmed in various cases [8, 13, 17]. Combined with Hitchin's result [29] mentioned above, we obtain a wealth of Kähler complex manifolds which do not admit non-trivial twisted generalized Kähler structures at all. As pointed out in [30], such examples include

(locally) deRham irreducible compact Kähler–Einstein manifolds with $c_1(M) < 0$ (see Theorem 6 below).

The existence of non-trivial generalized Kähler structures for which J_+ and J_- commute thus reduces to the following question:

Question 2. Let (M, J) be a compact complex manifold whose holomorphic tangent bundle splits as a direct sum of two holomorphic, integrable sub-bundles $T_\pm M$. Define a second almost complex structure J_- on M to be equal to J on T_-M and to $-J$ on T_+M. Does there exist a Riemannian metric g on M which is compatible with $J_+ := J$ and J_-, and such that $(g, J_\pm)$ is a generalized Kähler structure on M?

We note that the almost complex structure J_- defined as above is automatically integrable and commutes with J_+.

The fact that any maximal integral submanifold of $T_\pm M$ must be Kähler with respect to a compatible generalized Kähler metric quickly leads to non-Kähler examples where the answer to Question 2 is negative (see Example 1). Another obstruction comes from the fact that the fundamental 2-form of a compatible generalized Kähler metric must be $\partial\bar\partial$-closed (see Example 2). We are thus led to suspect that the above existence problem should be more tractable when (M, J) is of Kähler type, and we conjecture that in this case the answer to our Question 2 is '*yes*'. We are able to establish this in two special cases treated by Beauville in [8], namely when (M, J) admits a Kähler–Einstein metric (Theorem 5), and when (M, J) is four-dimensional ($m = 2$).

When M is four dimensional, our resuts are much sharper. In this case there are two classes of generalized Kähler structures, according to whether J_+ and J_- induce the same or different orientations on M. In this paper we shall refer to these cases as generalized Kähler structures of *bihermitian* or *ambihermitian* type, respectively, though in the terminology of [26] they would correspond to generalized Kähler structures of purely even and purely odd type, respectively. Note that generalized Kähler structures of ambihermitian type are precisely those for which J_+ and J_- commute and $J_+ \neq \pm J_-$.

In section 4, we solve completely the existence problem of generalized Kähler 4-manifolds of ambihermitian type, by proving the following result.

Theorem 1. *A compact complex surface (M, J) admits a generalized Kähler structure of ambihermitian type (g, J_+, J_-) with $J_+ = J$ if and only if the holomorphic tangent bundle of (M, J) splits as a direct sum of two holomorpic sub-bundles. Such a complex surface (M, J) is biholomorphic to one of the following:*

(a) *a geometrically ruled complex surface which is the projectivization of a projectively flat holomorphic vector bundle over a compact Riemann surface;*
(b) *a bi-elliptic complex surface, i.e. a complex surface finitely covered by a complex torus;*
(c) *a compact complex surface of Kodaira dimension* 1 *and even first Betti number, which is an elliptic fibration over a compact Riemann surface, whose only singular fibres are multiple smooth elliptic curves;*
(d) *a compact complex surface of general type, uniformized by the product of two hyperbolic planes $\mathbb{H} \times \mathbb{H}$ and with fundamental group acting diagonally on the factors.*

(e) *A Hopf surface, with universal covering space* $\mathbb{C}^2 \setminus \{(0,0)\}$ *and fundamental group generated by a diagonal automorphism* $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$ *with* $0 < |\alpha| \leq |\beta| < 1$*, and a diagonal automorphism* $(z_1, z_2) \mapsto (\lambda z_1, \mu z_2)$ *with* λ, μ *primitive* ℓ*-th roots of* 1.

(f) *An Inoue surface in the family* $S_{\mathcal{M}}$ *constructed in* [31].

On any of the above complex surfaces there exists a family (depending on one arbitrary smooth function on M*) of generalized Kähler structures of ambihermitian type.*

To prove this theorem we use the fact that the commuting complex structures give rise to a splitting of the holomorphic tangent bundle of (M, J_+) into two holomorphic line bundles $T_\pm M$. Using this splitting and the methods of [22], we describe the set of all generalized Kähler structures of ambihermitian type on such a complex surface. We thus establish a one-to-one correspondence between four-manifolds admitting generalized Kähler structures of ambihermitian type and complex surfaces with split holomorphic tangent bundle. The latter class of complex surfaces has been studied by Beauville [8]. We use his classification and some results from [50] to derive Theorem 1.

We further refine our classification by considering the *untwisted* case, i.e. when $[H] = 0 \in H^3(M, \mathbb{R})$, and the *twisted* case, where $[H]$ is nonzero. We show, by using the fundamental results of Gauduchon [22, 23], that untwisted generalized Kähler structures on compact four-manifolds can only exist when the first Betti number is even; likewise in the *twisted* case, any generalized Kähler 4-manifold must have odd first Betti number (Corollary 1).

2. Hermitian geometry

In this section we present certain key properties of Hermitian manifolds which we will need in the later sections, giving special attention to the four-dimensional case. Let M be an oriented $2m$-dimensional manifold. A *Hermitian structure* on M is defined by a pair (g, J) consisting of a Riemannian metric g and an integrable almost complex structure J, which are *compatible* in the sense that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. The Hermitian structure (g, J) is called *positive* if J induces the given orientation on M and *negative* otherwise.

The complex structure J induces a decomposition $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ of the complexified vectors into $\pm i$ eigenspaces, and hence defines the usual bi-grading of complex differential forms

$$\Omega^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}(M),$$

where we let J act on T^*M by $(J\alpha)(X) = -\alpha(JX)$, so that it commutes with the Riemannian duality between vectors and 1-forms: $(J\alpha)^\sharp = J\alpha^\sharp$.

The product structure $\wedge^2 J$ induces a splitting of the real 2-forms into ± 1 eigenspaces:

$$\Omega^2(M) = \Omega^{J,+}(M) \oplus \Omega^{J,-}(M),$$

whose complexification is simply $\Omega^{J,+}(M) \otimes \mathbb{C} = \Omega^{1,1}(M)$ and $\Omega^{J,-}(M) \otimes \mathbb{C} = \Omega^{2,0}(M) \oplus \Omega^{0,2}(M)$. Furthermore, the *fundamental 2-form* $F = gJ$, a real $(1,1)$-form of square-norm m, defines a g-orthogonal splitting $\Omega^{J,+}(M) = \mathcal{C}^\infty(M) \cdot F \oplus$

$\Omega_0^{J,+}(M)$. In this way we obtain the $U(m)$ irreducible decomposition of real 2-forms:

$$\Omega^2(M) = \mathcal{C}^\infty(M)\cdot F \oplus \Omega_0^{J,+}(M) \oplus \Omega^{J,-}(M).$$

On a positive Hermitian 4-manifold, the above $U(2)$ splitting of $\Omega^2(M)$ is compatible with the $SO(4)$ decomposition $\Omega^2(M) = \Omega^+(M) \oplus \Omega^-(M)$ into self-dual and anti-self-dual forms, as follows:

$$\Omega^+(M) = \mathcal{C}^\infty(M)\cdot F \oplus \Omega^{J,-}(M); \quad \Omega^-(M) = \Omega_0^{J,+}(M). \tag{1}$$

For a negative Hermitian structure the rôles of $\Omega^+(M)$ and $\Omega^-(M)$ in the above identifications are interchanged. Thus, on an oriented Riemannian four-manifold (M,g), we obtain the well-known correspondence between smooth sections in $\Omega^+(M)$ (resp. $\Omega^-(M)$) of square-norm 2 and positive (resp. negative) almost Hermitian structures (g,J). Whereas the existence of such smooth sections is a purely topological problem, the existence of integrable ones depends essentially on g. This is measured (at least at a first approximation) by the structure of the Weyl curvature tensor W, cf. [2, 44, 45].

The *Lee form* $\theta \in \Omega^1(M)$ of a Hermitian structure is defined by

$$dF \wedge F^{m-2} = \frac{1}{(m-1)}\theta \wedge F^{m-1}, \tag{2}$$

or equivalently $\theta = J\delta^g F$ where δ^g is the co-differential with respect to the Levi–Civita connection D^g of g. Since J is integrable, dF measures the deviation of (g,J) from a Kähler structure (for which J and F are parallel with respect to D^g). We have the following expression for D^gF (see e.g. [36, p.148]):

$$2g((D^g_X J)Y, Z) = d^cF(X, Y, JZ) + d^cF(X, JY, Z), \tag{3}$$

where $d^c = i(\bar\partial - \partial)$, so that $d^cF = \wedge^3 J(dF)$ is a real 3-form of type $(1,2)+(2,1)$.

In four dimensions, (2) reads as

$$dF = \theta \wedge F, \tag{4}$$

and (3) becomes (see e.g. [22, 49])

$$D^g_X F = \tfrac{1}{2}(X^\flat \wedge J\theta + JX^\flat \wedge \theta), \tag{5}$$

where $X^\flat = g(X)$ denotes the g-dual 1-form to X. We see from this that a Hermitian 4-manifold is Kähler if and only if $\theta = 0$.

The existence of a Kähler metric on a compact complex manifold (M^{2m}, J) implies the Hodge decomposition of the de Rham cohomology groups

$$H^k_{dR}(M,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(M),$$

where $H^{p,q}_{\bar\partial}(M)$ denote the Dolbeault cohomology groups. This, together with the equality $H^{p,q}_{\bar\partial}(M) \cong \overline{H^{q,p}_{\bar\partial}(M)}$, implies that the odd Betti numbers of a complex manifold admitting a Kähler metric must be even. When $m=2$, it turns out that this condition is also sufficient.

Theorem 2. [11, 38, 46, 48] *Let M be a compact four-manifold endowed with an integrable almost complex structure J. Then there exists a compatible Kähler metric on (M,J) if and only if $b_1(M)$ is even.*

This important result was first established by Todorov [48] and Siu [46], using the Kodaira classification of compact complex surfaces. Direct proofs were found recently by Buchdahl [11] and Lamari [38].

Since we deal with complex manifolds of non-Kähler type (i.e. do not admit any Kähler metric), we recall the definition of the $\partial\bar{\partial}$-cohomology groups:

$$H^{p,q}_{\partial\bar{\partial}}(M) := \{d\text{-closed } (p,q)\text{-forms}\}/\partial\bar{\partial}\{(p-1,q-1)\text{-forms}\}.$$

Note that there is a natural map

$$\iota : H^{p,q}_{\partial\bar{\partial}}(M) \to H^{p,q}_{\bar{\partial}}(M).$$

When (M,J) is of Kähler type, the well-known $\partial\bar{\partial}$-lemma (see e.g. [15]) states that the above map is in fact an isomorphism:

Proposition 1. ($\partial\bar{\partial}$-lemma) *If (M,J) is a compact complex manifold admitting a Kähler metric, then $\iota : H^{p,q}_{\partial\bar{\partial}}(M) \to H^{p,q}_{\bar{\partial}}(M)$ is an isomorphism.*

The $\partial\bar{\partial}$-lemma also holds on some non-Kähler manifolds, for example on all non-projective Moišezon manifolds. In fact, the $\partial\bar{\partial}$-lemma is preserved under bimeromorphic transformations and, therefore, holds on any compact complex manifold which is bimeromorphic to a Kähler manifold (i.e. is in the so-called *Fujiki class* $\mathcal{C}$), cf. [15].

While the existence of Kähler metrics on a compact complex manifold (M,J) is generally obstructed, a fundamental result of Gauduchon [22] states that on any compact conformal Hermitian manifold (M,c,J), there exists a unique (up to scale) Hermitian metric $g \in c$, such that its Lee form θ is co-closed, i.e. satisfies $\delta^g\theta = 0$. Such a metric is called a *standard* metric of c. By (2), a standard metric of (c,J) can be equivalently defined by the equation

$$2i\partial\bar{\partial}F^{m-1} = dd^c(F^{m-1}) = 0.$$

We now recall how, in four dimensions, the harmonic properties of the Lee form with respect to a standard metric are related the parity of the first Betti number (compare with Theorem 2 above).

Proposition 2. [22, 23] *Let M be a compact four-manifold endowed with a conformal class c of Hermitian metrics, with respect to an integrable almost complex structure J. Let g be a standard Hermitian metric in c. Then the following two conditions are equivalent:*

(i) *The first Betti number $b_1(M)$ is even.*
(ii) *The Lee form θ of g is co-exact.*

Proof. For the sake of completeness we outline a proof of this result. Let M be a compact four-manifold endowed with a standard Hermitian structure (g,J), and F and $\theta = J\delta^g F$ be the corresponding fundamental 2-form and Lee 1-form (with $\delta^g\theta = 0$).

We first prove that if $b_1(M)$ is even, then θ is co-exact (this is [22, Théorème II.1]). Applying the Hodge $*$ operator to θ, this is equivalent to showing that d^cF is exact. Recall that $2i\partial\bar{\partial}F = dd^cF = 0$ because g is standard. By Theorem 2, there exists a Kähler metric on (M,J) and then, by Proposition 1,

$$\bar{\partial}F = \partial\bar{\partial}\alpha,$$

for some $(0,1)$-form $\alpha = \xi - iJ\xi$. We deduce $d^cF = dd^c\xi$, as required.

In the other direction, we have to prove that if θ is co-exact then $b_1(M)$ is even. We reproduce an argument from [23]. With respect to a standard metric g, the forms θ and $J\theta = -\delta^g F$ are both co-closed, and therefore the $(0,1)$-form $\theta^{0,1} := \theta - iJ\theta$ is $\bar\partial$-coclosed. In terms of Hodge decomposition, this reads as

$$\theta^{0,1} = \theta^{0,1}_h + \bar\partial^*\Phi,$$

where $\Phi \in \Omega^{0,2}(M)$ and $\theta^{0,1}_h$ is the $(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$-harmonic part of $\theta^{0,1}$. Note that $\Phi = \alpha + i\beta$ where $\alpha, \beta \in \Omega^{J,-}(M)$ and $\alpha(\cdot,\cdot) := -\beta(J\cdot,\cdot)$.

We first claim that if $\theta^{0,1}_h = 0$, then $\phi = F + \beta$ is a harmonic self-dual 2-form. Indeed, since J is integrable, it satisfies $(D^g_{JX}J)(JY) = (D^g_X J)(Y)$ (see (3)), and therefore $J(\delta^g\beta) = \delta^g\alpha$, i.e.

$$\theta - iJ\theta = \bar\partial^*\Phi = \delta^g\Phi = \delta^g\alpha + i\delta^g\beta.$$

It follows that $J\theta = -\delta^g\beta$, and thus $\delta^g\phi = J\theta + \delta^g\beta = 0$.

By a well-known result of Kodaira (see e.g. [6]), a compact complex surface has even $b_1(M)$ if and only if the dimension $b_+(M)$ of the space of harmonic self-dual 2-forms on (M,g) is equal to $2h^{2,0}(M) + 1$, where $h^{2,0}(M) = \dim_{\mathbb{C}} H^{2,0}_{\bar\partial}(M)$; otherwise $b_+(M) = 2h^{2,0}(M)$. It follows that $b_1(M)$ is even if and only if $b_+(M) > 2\dim_{\mathbb{C}} H^{2,0}_{\bar\partial}(M)$.

Therefore, it suffices to show that $\theta^{0,1}_h = 0$, provided that θ is co-exact (because ϕ will be then a harmonic self-dual 2-form which is not a real part of a holomorphic $(2,0)$-form). To this end, we consider the natural map $\kappa : H^1_{dR}(M) \to H^{0,1}_{\bar\partial}(M) \cong H^1(M,\mathcal{O})$ from de Rham to Dolbeault cohomology given by $\xi \mapsto \xi^{0,1}$ on representatives. One easily checks that κ is well-defined and injective. Moreover, by the Noether formula (see e.g. [6]), κ is an isomorphism of (real) vector spaces if and only if $b_1(M)$ is even; otherwise, the image of $H^1_{dR}(M)$ in $H^{0,1}_{\bar\partial}(M)$ is of real codimension one.

For any element $\xi^{0,1} = \xi - iJ\xi$ in the image of κ, we calculate its L_2-hermitian product with $\theta^{0,1}_h$:

$$\begin{aligned}
\langle\theta^{0,1}_h, \xi^{0,1}\rangle_{L_2} &= \langle\theta^{0,1}, \xi^{0,1}\rangle_{L_2} - \langle\bar\partial^*\Phi, \xi^{0,1}\rangle_{L_2} \\
&= \langle\theta^{0,1}, \xi^{0,1}\rangle_{L_2} - \langle\Phi, \bar\partial\xi^{0,1}\rangle_{L_2} \\
&= \langle\theta^{0,1}, \xi^{0,1}\rangle_{L_2} = \frac{1}{2}(\theta,\xi)_{L_2} + \frac{i}{2}(J\theta,\alpha)_{L_2} \\
&= \frac{1}{2}(\theta,\xi)_{L_2} - \frac{i}{2}(\delta^g F,\alpha)_{L_2} = \frac{1}{2}(\theta,\xi)_{L_2}.
\end{aligned}$$

It follows that $\langle\theta^{0,1}_h, \xi^{0,1}\rangle_{L_2} = 0$, if θ is co-exact (because ξ is closed). Thus, in this case, the image of κ is contained in the complex subspace of $H^1_{\bar\partial}(M)$ which is orthogonal to $\theta^{0,1}_h$, and therefore would have real codimension at least 2, unless $\theta^{0,1}_h = 0$. $\square$

Finally, we review some natural connections which are useful in the Hermitian context. An integrable almost complex structure J induces a canonical holomorphic

structure on the tangent bundle TM, via the Cauchy–Riemann operator which acts on smooth sections X and Y of TM by

$$\bar{\partial}_X Y := \tfrac{1}{2}([X,Y] + J[JX,Y]) = -\tfrac{1}{2}J(\mathcal{L}_Y J)(X).$$

Identifying TM with the complex vector bundle $T^{1,0}M$, this operator may be viewed as a partial connection and has the equivalent expression

$$\bar{\partial}_X Y = [X,Y]^{1,0}, \tag{6}$$

for any complex vector fields X and Y of type $(0,1)$ and $(1,0)$, respectively.

In a similar way, any J-linear connection ∇ determines a partial connection $\bar{\partial}^{\nabla}$ on $T^{1,0}$ by projection, or acting on real vector fields by

$$\bar{\partial}^{\nabla}_X Y = \tfrac{1}{2}(\nabla_X Y + J\nabla_{JX} Y). \tag{7}$$

The operators $\bar{\partial}$ and $\bar{\partial}^{\nabla}$ have the same symbol but do not coincide in general. However, it is well-known that for any Hermitian structure (g,J), there exists a unique connection ∇, called the *Chern connection* of (g,J), which preserves both J and g, and such that $\bar{\partial}^{\nabla} = \bar{\partial}$. Note that the Chern connection ∇ has torsion, unless (g,J) is Kähler. It is related to the Levi–Civita connection D^g by (see e.g. [24]):

$$g(\nabla_X Y, Z) = g(D^g_X Y, Z) + \tfrac{1}{2}d^c F(X, JY, JZ). \tag{8}$$

In four dimensions, one uses (4) to rewrite (8) in the following form (cf. [22, 49]):

$$\nabla_X - D^g_X = \tfrac{1}{2}\big(X^\flat \otimes \theta^\sharp - \theta \otimes X + J\theta(X)J\big), \tag{9}$$

where $\theta^\sharp = g^{-1}(\theta)$ stands for the vector field g-dual to θ.

3. Generalized Kähler structures

As described in the introduction, a generalized Kähler structure on a manifold M consists of a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized complex structures such that $\langle \mathcal{J}_1 \,\cdot, \mathcal{J}_2 \,\cdot\rangle$ determines a definite metric on $TM \oplus T^*M$. The generalized complex structures $\mathcal{J}_1, \mathcal{J}_2$ are integrable with respect to the Courant bracket on sections of $TM \oplus T^*M$, given by

$$[X+\xi, Y+\eta]_H = [X,Y] + L_X\eta - L_Y\xi - \tfrac{1}{2}d(i_X\eta - i_Y\xi) + i_Y i_X H,$$

which depends upon the choice of a closed 3-form H, called the *torsion* or twisting. The space of 2-forms $b \in \Omega^2(M)$ acts on $TM \oplus T^*M$ by orthogonal transformations via

$$e^b(X+\xi) = X + \xi + i_X b,$$

and this action affects the Courant bracket in the following way

$$[e^b(W), e^b(Z)]_H = e^b[W,Z]_{H+db}.$$

So, if $(\mathcal{J}_1, \mathcal{J}_2)$ is integrable with respect to the H-twisted Courant bracket, then $(e^{-b}\mathcal{J}_1 e^b, e^{-b}\mathcal{J}_2 e^b)$ is integrable for the $(H+db)$-twisted Courant bracket.

A generalized complex structure $\mathcal{J}$, because it is orthogonal and squares to -1, lies in the orthogonal Lie algebra, and therefore may be decomposed according to the splitting

$$\mathfrak{so}(TM \oplus T^*M) = \wedge^2 TM \oplus \mathrm{End}(TM) \oplus \wedge^2 T^*M,$$

or, in block matrix form,

$$\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & A \end{pmatrix},$$

where π is a bivector field, A is an endomorphism of TM, and σ is a 2-form. Just as for an ordinary complex structure, the integrability of $\mathcal{J}$ may be expressed as the vanishing of a Nijenhuis tensor $[\mathcal{J},\mathcal{J}] = 0$ obtained by extending the Courant bracket. Restricted to $\wedge^2 TM$, this specializes to the usual Schouten bracket of bivector fields, requiring that $[\pi,\pi] = 0$. This means that π is a Poisson structure.

In [26], a complete characterization of the components of the generalized Kähler pair $(\mathcal{J}_1,\mathcal{J}_2)$ was given in terms of Hermitian geometry, which we now repeat here.

Theorem 3 ([26], Theorem 6.37)**.** *For any generalized Kähler structure $(\mathcal{J}_1,\mathcal{J}_2)$, there exists a unique 2-form b and Riemannian metric g such that*

$$e^{-b}\mathcal{J}_{1,2}e^{b} = \frac{1}{2}\begin{pmatrix} J_+ \pm J_- & -(F_+^{-1} \mp F_-^{-1}) \\ F_+ \mp F_- & J_+ \pm J_- \end{pmatrix},$$

where $J_\pm$ are integrable g-compatible complex structures and $F_\pm = gJ_\pm$ satisfy

$$d_+^c F_+ + d_-^c F_- = 0,\ dd_+^c F_+ = 0. \tag{10}$$

Conversely, any pair of g-compatible complex structures satisfying condition (10) define a generalized Kähler structure. Note that the pair $(\mathcal{J}_1,\mathcal{J}_2)$ is integrable with respect to the $(H-db)$-twisted Courant bracket where

$$H = d_+^c F_+.$$

An immediate corollary of this result and the preceding discussion is that the bivector fields

$$\pi_1 = -F_+^{-1} + F_-^{-1}, \qquad \pi_2 = -F_+^{-1} - F_-^{-1} \tag{11}$$

are both Poisson structures, a fact first derived in [42] directly from (10).

We also see from the theorem that by taking a bi-Hermitian structure (g, J_+, J_-) such that $J_+ = \pm J_-$, one obtains $d_+^c F_+ = d_-^c F_-$ and therefore (10) reduces to $d_+^c F_+ = 0$, which is nothing but the ordinary Kähler condition on (g, J_+).

As mentioned in the introduction, when $m > 2$ the second relation in (10) imposes a nontrivial constraint on the underlying complex manifolds $(M, J_\pm)$: they must admit a (common) Hermitian metric g for which the fundamental 2-forms are dd^c-closed. Furthermore, if the complex manifold (M, J_+) satisfies the $\partial\bar\partial$-lemma (see Proposition 1), then the torsion $H = d_+^c F_+$ of any compatible generalized Kähler structure must be exact.

Proposition 3. *Let (M,J) be a compact complex manifold such that $\iota : H^{1,2}_{\partial\bar\partial}(M) \to H^{1,2}_{\bar\partial}(M)$ is an isomorphism. Then any generalized Kähler structure on M is* untwisted*, i.e.* $[H] = 0$.

We now proceed with an investigation of the class of generalized Kähler structures (g, J_+, J_-) for which the pair of complex structures *commute* but are unequal, i.e. which satisfy $[J_+, J_-] = 0$ and $J_+ \neq \pm J_-$. In the following theorem, we show that the splitting

$$TM = T_+M \oplus T_-M,$$

determined by the ± 1-eigenbundles of $Q = J_+J_-$, is not only integrable, i.e. determines two transverse foliations of M, but is also holomorphic with respect to $J_\pm$, and that the leaves of each foliation inherit a natural Kähler structure.

Theorem 4. *Let (g, J_+, J_-) define a generalized Kähler structure with $[J_+, J_-] = 0$. Then the ± 1-eigenspaces of $Q = J_+J_-$ define g-orthogonal $J_\pm$-holomorphic foliations on whose leaves g restricts to a Kähler metric.*

Proof. Let $T_\pm M = \ker(Q \mp \mathrm{id}) = \ker(J_+ \pm J_-)$. Since $\ker(J_+ \pm J_-) = \mathrm{im}(J_+ \mp J_-)$, we see that $T_\pm M$ coincide with the images of the Poisson structures

$$\pi_1 = (J_+ - J_-)g^{-1}, \qquad \pi_2 = (J_+ + J_-)g^{-1}$$

from (11). Therefore $T_\pm M$ are integrable distributions and determine transverse foliations of M. Since Q is an orthogonal operator, we see further that the foliations defined by its ± 1 eigenvalues must be orthogonal with respect to the metric g.

The complex structures induce decompositions $T_+M \otimes \mathbb{C} = A \oplus \overline{A}$ and $T_-M \otimes \mathbb{C} = B \oplus \overline{B}$, where

$$A = T^{1,0}_{J_+}M \cap T^{0,1}_{J_-}M, \qquad B = T^{1,0}_{J_+}M \cap T^{1,0}_{J_-}M$$

are themselves integrable since they are intersections of integrable distributions. We now show that A is preserved by the Cauchy-Riemann operator of J_+, proving that T_+M is a J_+-holomorphic sub-bundle. Let X be a $(0,1)$-vector field for J_+ and let $Z \in C^\infty(A)$. Then

$$\overline{\partial}_X Z = [X, Z]^{1,0}.$$

Since $T^{1,0}_{J_+}M = A \oplus B$, we may project to these two components:

$$\overline{\partial}_X Z = [X, Z]_A + [X, Z]_B.$$

To show that A is J_+-holomorphic, we must show the vanishing of the second term, which upon expanding $X = X_{\overline{A}} + X_{\overline{B}}$, reads

$$[X, Z]_B = [X_{\overline{A}}, Z]_B + [X_{\overline{B}}, Z]_B.$$

The first term vanishes since $A \oplus \overline{A} = T_+M \otimes \mathbb{C}$ is involutive, and the second term vanishes since $A \oplus \overline{B} = T^{0,1}_{J_-}M$ is involutive. Therefore A is J_+-holomorphic. An identical argument proves that B is J_+-holomorphic, and that both A, B are J_--holomorphic, as required.

To show that g restricts to a Kähler metric on the leaves of $T_\pm M$, observe that since $J_+ = J_-$ along the leaves of T_-M, we have upon restriction $d^c_+F_+ = d^c_-F_-$. Similarly along the leaves of T_+M we have $J_+ = -J_-$, so that upon restriction, $d^c_+ = -d^c_-$ and $F_+ = -F_-$, giving again $d^c_+F_+ = d^c_-F_-$. But since the generalized Kähler condition forces $d^c_+F_+ = -d^c_-F_-$, we conclude that both $F_\pm$ are closed upon restriction to the leaves of either foliation, therefore defining Kähler structures there. $\square$

The holomorphicity of the decomposition $TM = T_+M \oplus T_-M$ proven above together with the condition $d^c_+F_+ + d^c_-F_- = 0$ also imply that Q is parallel with respect to the Chern connections $\nabla^\pm$ of $J_\pm$; in other words, for a generalized Kähler structure with $[J_+, J_-] = 0$, the Chern connections $\nabla^\pm$ have holonomy contained in $U(m_+) \times U(m_-)$ where $\dim_{\mathbb{R}} T_\pm M = 2m_\pm$. We now provide an alternative proof of this fact, avoiding the use of Theorem 3.

Proposition 4. *Let (J_+, J_-) be a pair of Hermitian complex structures for the Riemannian metric g, such that $d^c_+F_+ + d^c_-F_- = 0$ and $[J_+, J_-] = 0$. Then $Q = J_+J_-$ is covariant constant with respect to the Chern connections $\nabla^\pm$.*

Proof. Since $\nabla^+J_+ = 0$ by definition, it suffices to show that $\nabla^+J_- = 0$. From Equation (3), we see that

$$\nabla^+ - \nabla^- = L,$$

where $L \in \Omega^1(\mathrm{End}(TM))$ is given by

$$2g(L_X Y, Z) = d^c_+F_+(X, J_+Y, J_+Z) - d^c_-F_-(X, J_-Y, J_-Z).$$

Consequently, $\nabla^+J_- = \nabla^-J_- + [L, J_-]$. By definition, $\nabla^-J_- = 0$, and expanding the commutator we obtain

$$\begin{aligned} 2g([L_X, J_-]Y, Z) &= d^c_+F_+(X, J_+J_-Y, J_+Z) + d^c_-F_-(X, Y, J_-Z) \\ &\quad + d^c_+F_+\big(X, J_+Y, J_+J_-Z\big) + d^c_-F_-\big(X, J_-Y, Z\big). \end{aligned} \tag{12}$$

If Y is taken in T_+M and Z in T_-M, then the terms cancel since $d^c_+F_+ + d^c_-F_- = 0$. If $Y, Z \in T_+M$, then trivially $g((\nabla^+_X J_-)Y, Z) = g((\nabla^+_X J_+)Y, Z) = 0$ and similarly for $Y, Z \in T_-M$. Hence ∇^+J_- must vanish identically. Similarly, $\nabla^-J_+ = 0$, proving the result. □

In fact, this proposition provides an alternative proof not only of the holomorphicity of $T_\pm M$ but also of their integrability, by observing that since the torsion of ∇^+ vanishes upon restriction to $T_\pm M$, we have for $Y, Z \in T_+M$ or T_-M,

$$[Y, Z] = \nabla^+_Y Z - \nabla^+_Z Y,$$

and since $\nabla^+Q = 0$, $T_\pm M$ are involutive for the Lie bracket. Applying the same argument to T_-M, we obtain an alternative proof of Theorem 4.

Remark 1. Along the above lines one can establish the following result: Let J_+ and J_- be a pair of commuting almost complex structures on a $2m$-manifold M, such that J_+ is integrable, and let $T_\pm M$ denote the sub-bundles of TM corresponding to (± 1)-eigenspaces of $Q = J_+J_-$. Then any two of the following three conditions imply the third.

(a) $T_\pm M$ are integrable sub-bundles of TM;
(b) $T_\pm M$ are holomorphic sub-bundles of TM with respect to J_+;
(c) J_- is an integrable almost complex structure.

Let us now return to the existence problem. According to Theorem 4, we must consider complex manifolds (M, J) whose tangent bundle splits as a direct sum of two integrable, holomorphic sub-bundles $T_\pm M$; the second complex structure J_- is obtained from $J_+ = J$ by composing with Q, the product structure defining $T_\pm M$. It is then natural to ask whether there is a Riemannian metric g on M which is compatible with the commuting pair (J_+, J_-), satisfying the generalized Kähler condition. (This is Question 2 of the introduction.)

Locally, the answer is always '*yes*'. Indeed, by using complex coordinates adapted to the transverse foliations, i.e. a neighborhood $U = V \times W \subset \mathbb{C}^{m_1} \times \mathbb{C}^{m_2}$ such that $T_-U = TV$, $T_+U = TW$, then for any Kähler metrics g_V and g_W on V and W, the product metric $g_U := g_V \times g_W$ is Kähler with respect to both $J_\pm$, and $(g_U, J_\pm)$ is a generalized Kähler structure.

We now show that if there exists one generalized Kähler metric g on (M, J_+, J_-), then there is in fact a whole family parametrized by smooth functions (This is similar to the variation of a Kähler metric by adding $dd^c f$). This construction is closely related to the potential theory developed in [21, 41]. We will use the integrable decomposition

$$TM = T_+M \oplus T_-M,$$

and the associated decomposition $d = \delta_+ + \delta_-$ of the exterior derivative (induced by the 'type' decomposition $\wedge^* T^*M = (\wedge^* T_+M^*) \otimes (\wedge^* T_-M^*)$), so that, defining $\delta^c_\pm = [J_+, \delta_\pm]$, we have

$$d^c_\pm = \pm\delta^c_+ + \delta^c_-. \tag{13}$$

Proposition 5. *Let (M, g, J_+, J_-) be a generalized Kähler structure. Then, for any smooth function $f \in C^\infty(M, \mathbb{R})$ and sufficiently small real parameter t, the 2-form*

$$\tilde{F}_+ = F_+ + t(\delta_+\delta^c_+ f - \delta_-\delta^c_- f) \tag{14}$$

defines a new Riemannian metric $\tilde{g} = -\tilde{F}_+ J_+$ which is compatible with both $J_\pm$, and such that $(\tilde{g}, J_\pm)$ defines a generalized Kähler structure with unmodified torsion class $[H] \in H^3(M, \mathbb{R})$.

Proof. The $J_\pm$-invariant 2-form in (14) defines the J_--fundamental form $\tilde{F}_- = \tilde{g}J_-$, or

$$\tilde{F}_- = F_- + t(-\delta_+\delta^c_+ f - \delta_-\delta^c_- f).$$

We now show that $d^c_+\tilde{F}_+ + d^c_-\tilde{F}_- = 0$, since

$$\begin{aligned} d^c_+(\delta_+\delta^c_+ - \delta_-\delta^c_-) + d^c_-(-\delta_+\delta^c_+ - \delta_-\delta^c_-) &= -\delta^c_+\delta_-\delta^c_- + \delta^c_-\delta_+\delta^c_+ \\ &\quad + \delta^c_+\delta_-\delta^c_- - \delta^c_-\delta_+\delta^c_+ \\ &= 0. \end{aligned}$$

Finally, by the identity

$$\begin{aligned} d^c_+(\delta_+\delta^c_+ - \delta_-\delta^c_-) &= \delta^c_-\delta_+\delta^c_+ - \delta^c_+\delta_-\delta^c_- \\ &= (\delta_+ + \delta_-)\delta^c_+\delta^c_- \\ &= d\delta^c_+\delta^c_-, \end{aligned}$$

we see that $d^c_+(\tilde{F}_+ - F_+)$ is exact, showing that $[d^c_+F_+] = [d^c_+\tilde{F}_+]$, completing the proof. □

The following example shows that the global existence question is more subtle.

Example 1. Take the product $M = M_1 \times M_2$ of two complex manifolds (M_1, J_1) and (M_2, J_2), where the latter admits no Kähler metrics at all (see Theorem 2) and put $J_\pm := J_1 \pm J_2$ on $TM = TM_1 \oplus TM_2$. Then J_+ and J_- commute and induce the obvious holomorphic splitting of TM, but they cannot admit a compatible generalized Kähler metric g (see Theorem 4). In fact, (M, J_+, J_-) cannot admit any compatible Riemannian metric g with $d^c_+F_+ + d^c_-F_- = 0$ (see Proposition 4). Note that while (M, J) admits no Kähler metric, M_1 can be chosen so that (M, J) does admit Hermitian metrics with $\partial\bar{\partial}$-closed fundamental forms.

By contrast, if (M, J) is a complex manifold of Kähler type, we can always find a Riemannian metric compatible with both J_+ and J_- and such that $d^c_+ F + d^c_- F_- = 0$, as we now show.

Lemma 1. *Let (M, J_+) be a complex manifold of Kähler type whose tangent bundle splits as a direct sum of two holomorphic, integrable sub-bundles $T_\pm M$, and let $J_- = -J|_{T_+M} + J|_{T_-M}$. Then M admits a Riemannian metric g, compatible with both J_+ and J_-, satisfying $d^c_+ F_+ + d^c_- F_- = 0$.*

Proof. Let g_0 be any Kähler metric for (M, J_+); since $J_\pm$ commute, the J_--averaged Riemannian metric

$$g(\cdot,\cdot) := \tfrac{1}{2}(g_0(\cdot,\cdot) + g_0(J_-\cdot, J_-\cdot))$$

is compatible with both $J_\pm$. We claim that g has the desired properties.

To see this, decompose the original Kähler form F_0 according to the splitting $\wedge^2 T^*M = \wedge^2 T^*_+ M \oplus (T^*_+ M \otimes T^*_- M) \oplus \wedge^2 T^*_- M$, yielding

$$F_0 = F_{++} + F_{+-} + F_{--}.$$

Then the fundamental forms for $(g, J_\pm)$ are

$$F_\pm = \pm F_{++} + F_{--},$$

and using Equation (13) and the fact $dF_0 = 0$, we obtain

$$\begin{aligned} d^c_- F_- &= \delta^c_-(-F_{++}) - \delta^c_+ F_{--} \\ &= -d^c_+ F_+, \end{aligned}$$

as required. □

Note that in the above Lemma, the commuting bi-Hermitian structure $(g, J_\pm)$ is not necessarily generalized Kähler, because although $d^c_+ F_+ + d^c_- F_- = 0$, it is not necessarily the case that $dd^c_+ F_+ = 0$. We now provide an example where this final condition cannot be fulfilled.

Example 2. We elaborate on an example from [14] of a compact 6-dimensional solvmanifold M which does not admit a Kähler structure. M is obtained as a compact quotient of a complex 3-dimensional Lie group (biholomorphic to $\mathbb{C}^3$) whose complex Lie algebra $\mathfrak{g}$ is generated by the complex $(1,0)$-forms $\sigma_1, \sigma_2, \sigma_3$, such that

$$d\sigma_1 = 0, \; d\sigma_2 = \sigma_1 \wedge \sigma_2, \; d\sigma_3 = -\sigma_1 \wedge \sigma_3.$$

Thus, $\mathfrak{g}$ (and hence M) inherits a natural left-invariant complex structure J with respect to which the σ_i are holomorphic 1-forms. Note that (M, J) does not satisfy the $\partial\bar{\partial}$-lemma because σ_2 and σ_3 are holomorphic but not closed.

It is straightforward to check that there are no left-invariant Hermitian metrics g on $(\mathfrak{g}, J)$ such that the condition $dd^c F = 0$ is satisfied. Since the volume form $v = \sigma_1 \wedge \overline{\sigma}_1 \wedge \sigma_2 \wedge \overline{\sigma}_2 \wedge \sigma_3 \wedge \overline{\sigma}_3$ is bi-invariant, a standard argument [9, 19] shows that (M, J) does not admit *any* Hermitian metrics with dd^c-closed fundamental form. In particular, (M, J) admits no compatible generalized Kähler structures.

However, we can define a second left-invariant complex structure J_- on $\mathfrak{g}$ (and hence also on M) such that $T^{1,0}_{J_-} M = \mathrm{span}_{\mathbb{C}}\{\overline{\sigma}_1, \sigma_2, \sigma_3\}$, so that $J_+ := J$ and J_- are both integrable, commute and define *holomorphic* (and therefore integrable) sub-bundles $T_\pm M$. Furthermore, the left-invariant metric $g_0 = \sum_{i=1}^3 \sigma_i \otimes \overline{\sigma}_i$ on $\mathfrak{g}$

defines on M a Hermitian metric which is compatible with both J_+ and J_-, and such that $d^c_+ F_+ + d^c_- F_- = 0$.

For a compact complex manifold of Kähler type, (M, J), Beauville conjectures [8] that TM splits as a direct sum of two holomorphic integrable sub-bundles if and only if M is covered by the product of two complex manifolds $M_+ \times M_-$ on which the fundamental group of M acts *diagonally*, i.e. $\pi_1(M)$ acts on each $M_\pm$ and its action on the product is the diagonal action. In the case when there is a Kähler metric on (M, J) whose Levi-Civita connection preserves T_+M and T_-M, the conjecture follows by the de Rham decomposition theorem. It has also been confirmed in other cases [8, 13, 17]. We mention here the following partial result.

Theorem 5. [8, 34] *Let (M, J) be a compact complex manifold which admits a Kähler–Einstein metric g, and whose tangent bundle splits as a direct sum of two holomorphic sub-bundles $T_\pm M$. Then $T_\pm M$ are parallel with respect to the Levi-Civita connection. In particular, g is Kähler with respect to both $J_+ = J$ and $J_- = -J|_{T_+M} + J|_{T_-M}$, and therefore (M, J) admits generalized Kähler metrics compatible with J_+ and J_-.*

Proof. This is a standard Bochner argument. Let g be a Kähler–Einstein metric on (M, J). The vector bundle $E = \mathrm{End}(TM)$ is a Hermitian holomorphic bundle with unitary connection D induced by the Levi–Civita connection. The Ricci endomorphism of E is defined by

$$K(Q) = [R, Q],$$

where $R \in C^\infty(E)$ is the usual Ricci endomorphism of the tangent bundle and $Q \in C^\infty(E)$. Since g is Kähler–Einstein, $K \equiv 0$.

A section $Q \in C^\infty(E)$ is holomorphic if and only if $D''Q = 0$, where $D = D' + D''$ is the usual decomposition of D into partial connections. The classical *Bochner–Kodaira* identity (see e.g. [35, 15]) implies that for any *holomorphic* section Q of E,

$$\int_M ||D'Q||^2_g v_g = \int_M g(K(Q), Q) v_g = 0. \tag{15}$$

Thus, any holomorphic section of E must be parallel. Applying this to $Q = J_+J_-$, we see that $T_\pm M$ are parallel for the Levi–Civita connection. By the de Rham decomposition theorem, (M, g, J) must be then a local Kähler product of two Kähler–Einstein manifolds tangent to $T_\pm M$, respectively. The claim follows. □

To conclude this section, we wish to indicate that the methods of Theorem 4 and Proposition 4 can be used to prove *non-existence* results as follows. When J_+ and J_- do not commute, a direct computation using (12) shows that the commutator $P = [J_+, J_-]$ satisfies

$$\nabla^+_X P + J_+(\nabla^+_{J_+X} P) = 0,$$

provided that $d^c_+ F_+ + d^c_- F_- = 0$. It follows that for any generalized Kähler structure $(g, J_\pm)$, P defines a $J_\pm$-*holomorphic* bivector field $\pi = Pg^{-1}$. This fact was first established in [3] for the case $m = 2$, and by Hitchin [29] in general; the latter work also shows that P defines a $J_\pm$-holomorphic *Poisson structure*, a fact which follows from the fact that π_1, π_2 are Poisson structures (see Equation (11)). Therefore, if (M, J) does not carry a non-trivial holomorphic Poisson structure

(e.g. if $H^0(M, \wedge^2(TM)) = 0$), then for any generalized Kähler structure $(g, J_\pm)$ with $J_+ = J$, J_+ and J_- must commute. Then, by Theorem 4, non-trivial generalized Kähler structures do not exist unless the holomorphic tangent bundle of (M, J) splits. Using results of [8, 13, 17] one finds a wealth of projective complex manifolds such that $H^0(M, \wedge^2(TM)) = 0$ and TM does not split. This argument has been used in [30] to prove that a locally de Rham irreducible Kähler–Einstein manifold with $c_1(M) < 0$ does not admit any non-trivial generalized Kähler structure, thus establishing a partial converse of Theorem 5.

Theorem 6. [30] *Let (M, J) be a compact complex manifold of negative first Chern class. Then it admits a non-trivial generalized Kähler structure (g, J_+, J_-) with $J_+ = J$ if and only if the holomorphic tangent bundle of (M, J) splits. In this case, J_+ and J_- commute.*

Proof. By the Aubin–Yau theorem [4, 51], (M, J) admits a Kähler–Einstein metric of negative scalar curvature. A standard Bochner argument shows $H^0(M, \wedge^2(TM)) = 0$. By the preceding remarks, for any generalized Kähler structure (g, J_+, J_-) with $J_+ = J$, the complex structures must commute and the result follows from Theorem 5. □

4. Generalized Kähler four-manifolds

In dimensions divisible by four, generalized Kähler structures fall into two broad classes, defined by whether the complex structures $\pm J_+$ and $\pm J_-$ induce the same or different orientations on the manifold.

Definition 1. Let M be a manifold of dimension $4k$. A triple (g, J_+, J_-), consisting of a Riemannian metric g and two g-compatible complex structures $J_\pm$ with $J_+ \neq \pm J_-$, is called a *bihermitian* structure if J_+ and J_- induce the same orientation on M; otherwise, it is called *ambihermitian*. Similarly, an (am)bihermitian conformal structure is a triple (c, J_+, J_-), where $c = [g]$ is a conformal class of (am)bihermitian metrics.

In this section we will concentrate on the 4-dimensional case, where we have the following characterization of the generalized Kähler condition in terms of the Lee forms $\theta_\pm$.

Proposition 6. *Let $(g, J_\pm)$ be an (am)bihermitian structure on a four-manifold M. Then the condition $d^c_+ F_+ + d^c_- F_- = 0$ is equivalent to $\theta_+ + \theta_- = 0$ in the bihermitian case, and to $-\theta_+ + \theta_- = 0$ in the ambihermitian case. The condition $dd^c_+ F_+ = 0$ means that g is a standard metric, i.e. $\delta^g \theta_+ = 0$. The twisting $[H]$ vanishes if and only if $\theta_+ = \delta^g \alpha$ for $\alpha \in \Omega^2(M)$, i.e. the Lee form is co-exact.*

Proof. By (4), we have $d^c_\pm F_\pm = (J_\pm \theta_\pm) \wedge F_\pm$, so that

$$d^c_+ F_+ + d^c_- F_- = (J_+ \theta_+) \wedge F_+ + (J_- \theta_-) \wedge F_-. \tag{16}$$

Note that in the bihermitian case $F_+ \wedge F_+ = F_- \wedge F_-$ is twice the volume form v_g, whereas in the ambihermitian case $F_+ \wedge F_+ = -F_- \wedge F_- = 2v_g$. Therefore, applying the Hodge star operator $*$ to (16) and using the fact that $\delta^g = - * d *$ when acting on 2-forms, we obtain the result. □

As an immediate corollary of this result, together with Proposition 2, we obtain[1]:

Corollary 1. *Let M be a generalized Kähler 4-manifold. If the torsion class $[H] \in H^3(M,\mathbb{R})$ vanishes, then the first Betti number must be even (and hence M is of Kähler type); if $[H] \neq 0$ then the first Betti number must be odd.*

Bihermitian complex surfaces were studied in [1, 3, 16, 33, 44] and classified for even first Betti number in [3], where the classification of Poisson surfaces [5] is used, and existence is only partially proven. In fact, [3] provides enough to show that in this case, any bihermitian structure is conformal to a unique generalized Kähler structure, up to scale.

Proposition 7. *Let (c, J_+, J_-) be a bihermitian conformal structure on a compact four-manifold M with $b_1(M)$ even. Then there is a unique (up to scale) metric $g \in c$ such that (g, J_+, J_-) is generalized Kähler.*

Proof. By [3, Lemma 4], any standard metric g of (c, J_+) (which is unique up to scale [22]) is standard for (c, J_-) as well, and furthermore $\theta_+ + \theta_- = 0$. By Proposition 6, this is equivalent to the generalized Kähler condition. □

Some constructions of these bihermitian structures can be found in [3, 12, 29, 33, 39], and these prove existence on many (but not all) of these surfaces.

In the case where the first Betti number is odd, bihermitian structures have been studied in [1, 3, 16, 44]. It follows from the results there that M must be a finite quotient of $(S^1 \times S^3)\sharp k\overline{\mathbb{C}P}^2$, $k \geq 0$. It is no longer true in this case that the standard metric provides a generalized Kähler metric in all cases. To the best of our knowledge, the only known examples of generalized Kähler structures on 4-manifolds with $b_1(M)$ odd are given by standard metrics in the anti-self-dual bihermitian conformal classes described in [44].

We now turn to the ambihermitian case, where we establish a complete classification of generalized Kähler structures. We start with the following observation.

Lemma 2. *Let M be a four-manifold endowed with a pair (J_+, J_-) of almost complex structures inducing different orientations on M. Then, M admits a Riemannian metric compatible with both $J_\pm$ if and only if J_+ and J_- commute. In this case, the tangent bundle splits*

$$TM = T_+M \oplus T_-M \tag{17}$$

as an orthogonal direct sum of Hermitian complex line bundles defined as the ± 1-eigenbundles of $Q = J_+J_-$.

Proof. Let g be a Riemannian metric on M, compatible with J_+ and J_-. Fix the orientation on M induced by J_+. As discussed in § 2, the fundamental 2-forms F_+ and F_- are sections of $\Omega^+(M)$ and $\Omega^-(M)$, respectively. Since $\Omega^-(M)$ is in the $+1$-eigenspace of $\wedge^2 J_+$, F_- is J_+-invariant. Hence J_+ and J_- commute. The converse is elementary. □

The proof of the above lemma shows that the existence of commuting almost complex structures on a four-manifold is a purely topological problem (in fact, it

[1]Alternatively, this result follows from the generalized Hodge decomposition for generalized Kähler structures proven in [27].

is equivalent to the existence of a field of oriented two-planes [43]). Note that a similar existence problem for pairs of *integrable* almost complex structures on M inducing different orientations was raised in [7], and has been almost completely solved in [37].

Our next step is to identify the compact complex surfaces (M, J) that admit a generalized Kähler metric (g, J_+, J_-) of ambihermitian type with $J_+ = J$.

Lemma 3. *Let (g, J_+, J_-) be an ambihermitian structure on a four-manifold M and let $Q = J_+J_-$ be the almost product structure it defines. Then the Lee forms satisfy $\theta_+ = \theta_-$ if and only if $T_\pm M$ are holomorphic sub-bundles for $J_\pm$, i.e. $\nabla^\pm Q = 0$. Then the standard metric in the conformal class defines a generalized Kähler metric.*

As a result, any compact complex surface (M, J) whose tangent bundle splits as a sum of holomorphic line bundles admits a compatible generalized Kähler metric.

Proof. If $\theta_+ = \theta_-$, then by Proposition 6, we have $d^c_+ F_+ + d^c_- F_- = 0$, and so $T_\pm M$ are holomorphic by Proposition 4.

In the other direction, we use Equation (9) and the fact that J_- is skew-symmetric to express

$$\nabla^+_X J_- = D^g_X J_- - \tfrac{1}{2}(X^\flat \wedge (J_-\theta_+)^\sharp + (J_-X)^\flat \wedge \theta_+^\sharp),$$

where $\alpha \wedge X = \alpha \otimes X - X^\flat \otimes \alpha^\sharp$ for $\alpha \in T^*M$ and $X \in TM$. Finally, by (5), we obtain

$$\nabla^+_X J_- = \tfrac{1}{2}(X^\flat \wedge J_-(\theta_- - \theta_+)^\sharp + J_-X^\flat \wedge (\theta_- - \theta_+)^\sharp). \tag{18}$$

It is clear from Equation (18) that $\nabla^+ J_-$ (and hence $\nabla^\pm Q$) vanishes if and only if $\theta_+ = \theta_-$, proving the result.

To prove the final statement, we note that any holomorphic one-dimensional sub-bundle $T_\pm M \subset TM$ is automatically integrable, and therefore the almost complex structure $J_- = -J|_{T_+M} + J|_{T_-M}$ is integrable. By definition, $J_+ = J$ and J_- commute, and $J_\pm$ induce different orientations. Clearly there are Riemannian metrics compatible with both $J_\pm$. Then we may apply the first part of the lemma. ☐

Now we are ready to prove our classification results for ambihermitian generalized Kähler structures.

Proof of Theorem 1. Let (M, g, J_+, J_-) be a compact generalized Kähler four-manifold of ambihermitian type. By Proposition 6 and Lemma 3, the holomorphic tangent bundle of (M, J_+) must split as a direct sum of two holomorphic line bundles $(T_\pm M, J_+)$. Complex surfaces with split tangent bundles were studied and essentially classified by Beauville [8]. We use his results to retrieve the list (a)–(f).

When $b_1(M)$ is even, the cases that occur according to [8] correspond to the surfaces listed in (a)–(d) of Theorem 1, modulo the fact that our description of the surfaces in (a) is slightly different from the one in [8, §5.5], and that the existence of a splitting of TM on *any* surface in (c) is not addressed in [8, §5.2].

To clarify these points, we notice that in the case of a ruled surface $M = P(E) \to \Sigma$, [8, Thm.C] implies that the universal cover is the product $\mathbb{C}P^1 \times \mathbb{U}$, where $\mathbb{U}$ is the universal covering space of Σ, and the diagonal action of $\pi_1(M) = \pi_1(\Sigma)$ gives rise to a $PGL(2, \mathbb{C})$ representation of $\pi_1(\Sigma)$, i.e. the holomorphic bundle E is projectively-flat as claimed in (a).

Note that for any an elliptic fibration $f : M \to \Sigma$ as in (c), the base curve Σ can be given the structure of an orbifold with a $2\pi/m_i$ cone point at each point corresponding to a fibre of multiplicity m_i (see, [8, § 5.2] and [50, § 7]). Since the Kodaira dimension of M is equal to 1, the orbifold Euler characteristic of Σ must be negative, and therefore Σ is a good orbifold uniformized by the hyperbolic space $\mathbb{H}$. Since the first Betti number of M is even, it follows from [50, Thm.7.4] the universal covering space of M is $\mathbb{C} \times \mathbb{H}$, on which the fundamental group $\pi_1(M)$ acts diagonally by isometries of the canonical product Kähler metric.

When $b_1(M)$ is odd, the possible cases are described in [8, §§ (5.2),(5.6),(5.7),(5.8)]. To prove that the only complex surfaces that really occur are those listed in (e) and (f) in Theorem 1 we have to exclude the possibility that (M, J_+) is an elliptic fibration of Kodaira dimension 1, odd $b_1(M)$, and with only multiple singular fibres with smooth reduction. It is shown in [8, § (5.2)] that for the holomorphic tangent bundle of such a surface to split, it must be covered by a product of simply connected Riemann surfaces on which the fundamental group acts diagonally. On the other hand, any elliptic surface M with Kodaira dimension 1 and $b_1(M)$ odd is finitely covered by an elliptic fiber bundle M' over a compact Riemann surface of genus > 1, which has trivial monodromy (cf. [50, p.139]). Since M (and hence M') is not Kähler, $b_1(M')$ is odd too. Wall [50, p.141] showed that the universal cover of such an M' is $\mathbb{C} \times \mathbb{H}$ on which $\pi_1(M')$ does not act diagonally. It then follows from Beauville's result cited above that the holomorphic tangent bundle of M' (and hence of M) does not split.

It remains to establish the existence of generalized Kähler metrics on the complex surfaces listed in Theorem 1. We know by Lemma 3 that there are ambihermitian metrics (g, J_+, J_-) on M, compatible with the holomorphic splitting of TM, which are parametrized by the choice of Hermitian metrics on each of the factors $T_\pm M$, or equivalently by two smooth functions on M. For any such metric g, we have $\theta^g_+ = \theta^g_-$, where $\theta^g_\pm$ are the corresponding Lee forms (see Lemma 3). Let g_0 be a standard metric of $([g], J_+)$, i.e. a metric in the conformal class $[g]$ such that $\delta^{g_0}(\theta^{g_0}_+) = 0$. Since $\theta^{g_0}_+ = \theta^{g_0}_-$, the triple (g_0, J_+, J_-) defines a generalized Kähler structure of ambihermitian type. Finally, since the standard metric is unique up to scale in any conformal class [22], we eventually obtain a family of generalized Kähler metrics on M, which depend on one arbitrary smooth function, completing the proof.

Remark 2. Some Hopf surfaces described in case (e) of Theorem 1 (e.g. those with $\alpha = \beta \in \mathbb{R}$ and $\lambda = \mu = 1$) admit a Riemannian metric g compatible with a pair of hyper-complex structures, $\mathcal{H}C_+$ and $\mathcal{H}C_-$, inducing different orientations on M, and such that for any choice $J_+ \in \mathcal{H}C_+$ and $J_- \in \mathcal{H}C_-$, (g, J_+, J_-) is a twisted generalized Kähler structure of ambihermitian type. Such Hopf surfaces do also admit an abundance of twisted generalized Kähler structures of bihermitian type [3, 44].

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